Superfast convergence effect in large orders of the perturbative and ε expansions for the $O(N)$-symmetric ϕ^4 model.

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Usually the asymptotic behavior for large orders of the perturbation theory is reached rather slowly. However, in the case of the N-component ϕ^4 model in $D = 4$ dimensions one can find a special quantity that exhibits an extremely fast convergence to the asymptotic form. A comparison of the available 5-loop result for this quantity with the asymptotic value shows agreement at the 10^{-3} level. An analogous superfast convergence to the asymptotic form happens in the case of the $O(N)$-symmetric anharmonic oscillator where this convergence has inverse factorial type. The large orders of the ε expansion for critical exponents manifest a similar effect.

I. INTRODUCTION

The series of the perturbation theory $\sum_k a_k g^k$ usually have factorially growing coefficients [1, 2, 3, 4]:

$$a_k = ck! A^k k^b \left[1 + O(k^{-1})\right] . \quad (1)$$

One of the problems with these series (see reviews [5, 6, 7] and references therein) is the slow convergence of the coefficients a_k to asymptotic form (1). In particular, for the $O(N)$-symmetric ϕ^4 model,

$$L = \frac{1}{2} \sum_{a=1}^{N} (\partial_\mu \phi_a)^2 + \frac{1}{4!} g \left(\sum_{a=1}^{N} \phi_a^2 \right)^2 , \quad (2)$$

the agreement between available 5-loop results [8] and asymptotic formulas (1) is rather poor, which may be attributed to the large $O(k^{-1})$ correction [9].

The aim of this paper is to attract attention to some *special quantities* whose asymptotic behavior is reached much faster than the typical slow $O(k^{-1})$ convergence in eq. (1). In particular, it will be shown that the 5-loop β function of the N-component ϕ^4 model contains some properties which agree with the asymptotic behavior with relative accuracy better than 10^{-3}.

In the case of the perturbative β function of the $O(N)$-symmetric ϕ^4 model $\beta(g, N) = \sum_{k=2}^{\infty} \beta_k(N) g^k$, the large-$k$ asymptotic behavior of $\beta_k(N)$ is given by

$$\beta_k(N) \overset{k \to \infty}{\longrightarrow} \beta_k^{as}(N) = c_N k! \left(-16\pi^2\right)^{-k} k^{\frac{N}{2}+3} \quad (3)$$

with c_N computed in Ref. [4] and simplified in Refs. [9, 10, 11]:

$$c_N = \left(\frac{9}{2\pi}\right)^{\frac{N-1}{6}} \exp\left\{(N + 2) \left[\frac{\zeta'(2)}{\pi^2} - \frac{\gamma}{6} - \frac{1}{4}\right] - 3\right\}$$

$$\times 96^{3/2} \frac{(N + 8)\Gamma\left(\frac{5}{2}\right)}{9\Gamma\left(\frac{N}{2} + 2\right)} \int_0^\infty dx \, x^{(N+17)/3} \left[K_1(x)\right]^4 . \quad (4)$$

Instead of the direct comparison of $\beta_k^{as}(N)$ with $\beta_k(N)$ (which shows rather poor agreement [12]) we choose another way. At $k \geq 3$, $\beta_k(N)$ is a polynomial in N of

k	$N_{k,r}$
3	-4.66667
4	$-4.02495, -41.3989$
5	$-4.01968, -12.0757, 3218.75$
6	$-4.00173, -8.75514, -44.0331, 504.74$

TABLE I: Roots $N_{k,r}$ of polynomials $\beta_k(N)$.

degree $k - 2$ and has $k - 2$ roots

$$\beta_k(N_{k,r}) = 0 \quad (1 \leq r \leq k - 2) . \quad (5)$$

Using the 5-loop β function of the N-component ϕ^4 model in the MS scheme [8], one can easily compute these roots numerically. The values are listed in Table I.

In this set of roots one can clearly see a magic sequence

$$\{-4.6667, -4.0249, -4.0197, -4.0017, \ldots\} \to -4 \quad (6)$$

which seems to converge very fast to the value -4. The aim of this paper is to show that we deal with a rather interesting effect of *superfast convergence* which is closely related to the *factorial divergence* of large-order perturbative coefficients (1), (3).

In Sec. II we will study a similar phenomenon in a much simpler model of the $O(N)$-symmetric anharmonic oscillator. This model will allow us to understand the origin of the superfast convergence and its relation to the properties of asymptotic expressions at the nonphysical point $N = -4$. In Sec. III we return to the ϕ^4 model and show that the situation is quite similar. In Sec. IV an analogous effect of superfast convergence is checked in large orders of the ε expansion for critical exponents.

II. ANHARMONIC OSCILLATOR

The N-component anharmonic oscillator

$$H_N = \frac{1}{2} \sum_{a=1}^{N} \left(p_a^2 + x_a^2\right) + g \left(\sum_{a=1}^{N} x_a^2 \right)^2 \quad (7)$$

k	$\nu_{k,1}(\nu_{k,1}^*)$	k	$\nu_{k,1}$
5	$-3.22834 \pm i0.426293$	12	$-4 + 0.0017843$
6	-3.44545	13	$-4 - 0.00027422$
7	$-3.63083 \pm i0.34226$	14	$-4 + 0.00003787$
8	-3.76443	15	$-4 - 4.8625 \times 10^{-6}$
9	$-3.9583 \pm i0.226557$	...	
10	$-4 + 0.0423159$	30	$-4 + 2.61370 \times 10^{-22}$
11	$-4 - 0.0123126$	31	$-4 - 1.53497 \times 10^{-23}$

TABLE II: Roots $\nu_{k,1}$ of polynomials $E_k(N)$ approaching the value -4.

and the properties of the perturbative expansion of the energy of the ground state

$$E(g, N) = \sum_{k=0}^{\infty} E_k(N)g^k \tag{8}$$

are well studied [1, 2, 7, 13, 14, 15]. $E_k(N)$ is a polynomial of degree $k + 1$ with $k + 1$ roots $\nu_{k,r}$:

$$E_k(\nu_{k,r}) = 0. \tag{9}$$

The roots close to -4 are listed in Table II, we label them with $r = 1$. They were computed numerically [16] using methods of Refs. [1, 2, 13, 14]. At $k \geq 10$, the roots $\nu_{k,1}$ are real and exhibit a very fast *inverse factorial convergence* to -4 which is described by the asymptotic formula derived in Appendix (see also Ref. [16]):

$$\nu_{k,1} + 4 \stackrel{k\to\infty}{\longrightarrow} 12\sqrt{2\pi}\frac{(-1)^k}{k!}3^{k/2}k^{3/2}. \tag{10}$$

The explanation of this sequence of roots tending to -4 is simple. The large-k asymptotic formula for $E_k(N)$ with the $O(k^{-1})$ correction [14] is

$$E_k(N) \stackrel{k\to\infty}{=} (-1)^{k+1}\Gamma\left(k + \frac{N}{2}\right)3^{k+\frac{N}{2}}\frac{2^{N/2}}{\pi\Gamma(N/2)}$$
$$\times \left[1 - \frac{1}{6k}\left(\frac{5}{3} + \frac{9}{2}N + \frac{7}{4}N^2\right) + O(k^{-2})\right]. \tag{11}$$

This asymptotic formula contains a factor of $[\Gamma(N/2)]^{-1}$ that has zeros at $N = 0, -2, -4\ldots$ At $k \geq 1$, all polynomials $E_k(N)$ have a factor of $N(N + 2)$ so that at we have common zeros of $[\Gamma(N/2)]^{-1}$ and $E_k(N)$ at $N = 0, -2$. The zero of $[\Gamma(N/2)]^{-1}$ at $N = -4$ corresponds to the *fast convergent sequence of zeros* of $E_k(N)$: $\nu_{r,1} \to -4$. One can wonder what happens with large zeros $N = -6, -8, \ldots$ of the function $[\Gamma(N/2)]^{-1}$ appearing on the RHS of (11). It can be checked numerically that at larger values of k polynomials $E_k(N)$ get additional series of roots convergent to the points $N = -6, -8, \ldots$. As an example, the zeros of the polynomial $E_{30}(N)$ are shown in Fig. 1 where the roots approaching points $N = -6, -8$ are clearly seen.
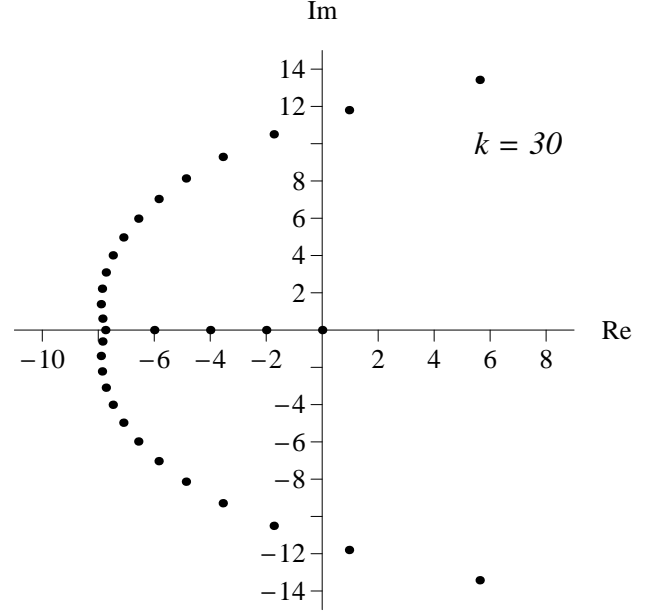

FIG. 1: Distribution of roots $\nu_{30,r}$ of polynomial $E_{30}(N)$ in the complex plane. The roots approaching the negative even points $-4, -6, -8$ are clearly seen as well as the stable roots 0 and -2.

III. ORIGIN OF THE ROOTS CONVERGENT TO -4 IN THE ϕ^4 MODEL

Now we return to the field theory. Expression (4) contributes a factor of $\left[\Gamma\left(2 + \frac{N}{2}\right)\right]^{-1}$ to $\beta_k^{as}(N)$. Therefore $\beta_k^{as}(N)$ vanishes at $N = -4, -6, \ldots$ (starting from $N = -8$ we must take into account the divergence of the integral containing $[K_1(x)]^4$ and the extra factor of $(N + 8)$). These zeros of $\beta_k^{as}(N)$ play the same role as zeros of $[\Gamma(N/2)]^{-1}$ in eq. (11) for the anharmonic oscillator. The zeros close to $N = -4$ appear already in the lowest orders of the β function as we saw in (6).

Strictly speaking, asymptotic behavior (3), (4) was derived for the function β in the MOM scheme whereas the 5-loop β function was computed in Ref. [8] in the MS scheme. The question about the asymptotic behavior of the perturbation theory for the β function in the MS scheme was considered in Ref. [5]. The MOM β function is available only in four loops [12]. For the MOM β function, we find a similar sequence of roots for 2, 3 and 4 loops $\{-4.66667, -4.09365, -4.12669\}$. We see a proximity to -4 but in the 4-loop MOM case the convergence is seen less clearly than in the 5-loop MS sequence (6).

One can trace the origin of the factor $[\Gamma(2 + (N/2))]^{-1}$ in the derivation of asymptotic expressions (3), (4) in Ref. [4]. In a more general context of the $O(N)$-symmetric $(\phi^2)^M$ model in $D = 2M/(M - 1)$ dimensions this factor has the form $[\Gamma(M + (N/2))]^{-1}$ and comes from the integration over the $O(N)$ collective co-

k	$q_{\omega,k,1}$	$q_{\nu^{-1},k,1}$	$q_{\eta,k,1}$
2	-4.66667	-3.38462	
3	-3.96014	-3.69220	-4.49615
4	-4.02773	-3.99515	-3.79616
5	-3.99610	-3.98743	-4.04522

TABLE III: Roots $q_{z,k,1} \rightarrow -4$ of polynomials $P_k^z(N)$ for critical exponents $z = \omega, \nu^{-1}, \eta$.

ordinates.

IV. ε EXPANSION

A similar effect of the fast convergence of roots to the value -4 can be found in large orders of the ε expansion for critical exponents. Using a compact notation for the critical exponents $z = \omega, \nu^{-1}, \eta$ of the universality class associated with the $O(N)$ symmetry, one can write the ε expansion in the form

$$z(\varepsilon, N) = \sum_k \left[\varepsilon (N+8)^{-2} \right]^k P_k^z(N). \qquad (12)$$

Here $P_k^z(N)$ are polynomials of N. Using expressions for $P_k^z(N)$ with $k \leq 5$ from Ref. [8], we can easily find the roots

$$P_k^z(q_{z,k,r}) = 0 \qquad (13)$$

tending to the value -4. These roots (labeled with $r = 1$) are listed in Table III. The convergence to the value -4 is clearly seen. This convergence is especially impressive for the critical exponent ω. Remember that this critical exponent is controlled by the β function only: $\omega = 2\beta_\varepsilon'(g_0)$, $\beta_\varepsilon(g_0) = 0$. Therefore there must be a certain correlation with the extremely good convergence that we saw for the roots associated with the coefficient functions of the β function in eq. (6).

The appearance of the same limit value -4 in the case of the ε expansion of critical exponents can be easily explained by the asymptotic formulas derived in Ref. [11]:

$$P_k^z(N) = -\bar{C}_z k! \left[-3 (N+8) \right]^k k^{(N+m_z)/2} \left[1 + O\left(\frac{\ln k}{k} \right) \right] \qquad (14)$$

where

$$m_\eta = 6, \quad m_{\nu^{-1}} = 8, \quad m_\omega = 10, \qquad (15)$$

$$\bar{C}_{\nu^{-1}} = 3\bar{C}_\eta, \quad \bar{C}_\omega = \frac{3(N+8)}{N+2}\bar{C}_\eta, \qquad (16)$$

$$\bar{C}_\eta = \frac{2^{-(N-16)/6} 3^{(N+3)/2} \pi^{-(N+8)/6}}{(N+8)\Gamma\left(\frac{N}{2}+1\right)} \exp\left[\frac{(N+2)\zeta'(2)}{\pi^2} \right.$$
$$\left. -\frac{1}{2}(N+6)\gamma - \frac{1}{4}(N+14) - \frac{3N+14}{N+8} \right]. \qquad (17)$$

Here we have the common factor $\left[\Gamma\left(\frac{N}{2}+1 \right) \right]^{-1}$ that has zeros at $N = -2, -4, -6, \ldots$ The zero $N = -2$ is trivial:

1) in the case of the critical exponent ω, this zero cancels in expression (16) for $\bar{C}_\omega$,

2) in the case of critical exponents ν^{-1}, η the factor $(N+2)$ is explicitly present in all orders of the ε (apart from the ε^0 contribution to η).

The zero at $N = -4$ is more interesting. It is responsible for the convergent sequences $q_{z,k,1} \xrightarrow{k \rightarrow \infty} -4$.

V. CONCLUSIONS

In spite of the widespread skepticism about relevance of the large-order asymptotic formulas for the current multi-loop calculations, we found quantities that reach the asymptotic behavior with high precision in rather low orders of the perturbation theory. The main idea was to turn from the traditional analysis of the perturbation theory at fixed N to the roots of the N-dependence.

Several manifestations of the superfast convergence of these roots to the asymptotic were studied: anharmonic oscillator, perturbative beta function of the ϕ^4 model and ε expansion for critical exponents. In the case of the anharmonic oscillator we could explicitly show that the convergence of roots to the asymptotic value -4 is factorially fast. In the ϕ^4 model the situation is less clear: the argument for the fast convergence of roots comes from the vanishing coefficient in asymptotic formula (4) at $N = -4$. It is rather astonishing that for the β function and for the critical exponent ω (directly related to the β function) we have 10^{-3} agreement with the asymptotic value already in the 5-th loop. Note that in the case of the anharmonic oscillator the convergence to the asymptotic form starts later. This may be related to the fact that we have complex roots in low orders of the perturbation theory for the anharmonic oscillator whereas in the ϕ^4 field theory the roots are real from the very beginning.

Although the superfast convergence deals with non-physical values $N \rightarrow -4$, this effect may still have useful applications:

i) The proximity of the N-roots to -4 can be used as an error test in multi-loop calculations.

ii) In field theory where the theoretical status of asymptotic formulas is sometimes not reliable, the effect of the superfast convergence may play the role of an additional control of theoretical assumptions.

iii) One could think about extensions of the method to other theories with a polynomial dependence on various parameters.

APPENDIX A: ASYMPTOTIC ESTIMATES FOR THE ANHARMONIC OSCILLATOR

1. Large-k asymptotic formula for the roots $\nu_{k,r}$

Let us derive asymptotic formula (10) for the roots of equation $E_k(\nu_{k,1}) = 0$. In the limit $k \to \infty$, $\nu_{k,1} \to -4$ we can solve this equation using a modified version of asymptotic formula (11) for the double limit $k \to \infty$, $N \to -4$:

$$E_k(N) \to (-1)^{k+1}\Gamma\left(k + \frac{N}{2}\right)3^{k+\frac{N}{2}}\frac{2^{N/2}}{\pi\Gamma\left(\frac{N}{2}\right)} + E_k(-4).$$ (A1)

We dropped the $1/k$ correction present in eq. (11) but added the extra term $E_k(-4)$ which becomes essential at $N \to -4$. At $N \to -4$ we can simplify eq. (A1)

$$E_k(N) \to (-1)^{k+1}\frac{k!}{k^3}\frac{3^{k-2}}{4\pi}(N+4) + E_k(-4).$$ (A2)

Now we find the roots of $E_k(N)$:

$$\nu_{k,1} + 4 \overset{k\to\infty}{=} (-1)^k\frac{4\pi}{3^{k-2}}\frac{k^3}{k!}E_k(-4).$$ (A3)

Inserting the asymptotic expression for $E_k(-4)$ from eq. (A8) derived below, we obtain asymptotic formula (10).

2. Large-k behavior of $E_k(-4)$

Now we want to derive an asymptotic formula for $E_k(-4)$ at large k. At even negative N the problem of the anharmonic oscillator (7) simplifies drastically. As was shown in Ref. [17] (see also [16]), energy $E(g,-4)$ is described by the cubic equation

$$\mathcal{P}[E(g,-4)] = 0, \quad \mathcal{P}(E) \equiv E^3 - 4E - 16g.$$ (A4)

Series (8) for $E(g,-4)$ has a finite radius of convergence which is determined by the singularity g_0 of $E(g,-4)$ closest to the point $g = 0$. This singularity comes from degenerate roots of cubic equation (A4):

$$\mathcal{P}(E_0) = \mathcal{P}'(E_0) = 0,$$ (A5)
$$E_0 = -2/\sqrt{3}, \quad g_0 = 3^{-3/2}.$$ (A6)

In the vicinity of this singular point

$$E(g,-4) \overset{g\to g_0}{=} E_0 - \sqrt{8\cdot 3^{-1/2}(g_0-g)}.$$ (A7)

The small-g expansion of this expression determines the large-k behavior

$$E_k(-4) \overset{k\to\infty}{\longrightarrow} \frac{1}{3}\sqrt{\frac{2}{\pi}}3^{3k/2}k^{-3/2}.$$ (A8)

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